A strong comparison result for viscosity solutions to Hamilton-Jacobi-Bellman equations with Dirichlet condition on a non-smooth boundary and application to parabolic problems

Sébastien Chaumont [a]

[a] *Institut Élie Cartan, Université Henri Poincaré Nancy I, B. P. 239, 54506 Vandoeuvre-lès-Nancy Cedex, France*

Abstract

In the framework of viscosity solutions, we give an extension of the strong comparison result for Hamilton-Jacobi-Bellman (HJB) equations with Dirichlet boundary conditions to the case of some non-smooth domains. In particular, it may be applied to parabolic problems on cylindrical domains.

Résumé

Principe de comparaison fort pour les solutions de viscosité de l'équation d'Hamilton-Jacobi-Bellman avec condition de Dirichlet sur une frontière irrégulière et application aux problèmes paraboliques. Dans le cadre de la théorie des solutions de viscosité, on donne une extension du principe de comparaison fort pour l'équation d'Hamilton-Jacobi-Bellman (HJB) avec condition au bord de type Dirichlet au cas de certains domaines irréguliers. En particulier, ce résultat est applicable aux problèmes paraboliques posés dans des domaines cylindriques.

Mathematics subject classifications (MSC 2000) : 49L25, 49N90, 35G25, 35G30, 35K35, 35K55, 35K60, 35K65.

1. Introduction

We study the Dirichlet problem for the following second-order degenerate Hamilton-Jacobi-Bellman (HJB) equation, arising in stochastic optimal control with exit time [1] :

$$\begin{cases} \sup_{\alpha\in A} \{-L^\alpha u(x) + c(x,\alpha)u(x) - f(x,\alpha)\} = 0 & \text{in } \Omega, \\ u(x) = \varphi(x) & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Email address: chaumont@iecn.u-nancy.fr (Sébastien Chaumont).

[1] See for instance Lions [11], Krylov [10] for a general presentation.

Ω is an open subset of $\mathbb{R}^d$ (with $d \geq 1$). The solution u is a real-valued function defined on $\overline{\Omega}$. $(L^\alpha)_\alpha$ is a family of linear elliptic operators indexed by a parameter α taking its values in a compact separable metric space A, defined by

$$L^\alpha \psi(x) = b(x,\alpha).D\psi(x) + \frac{1}{2}\text{trace}\left\{\sigma(x,\alpha)\sigma^*(x,\alpha)D^2\psi(x)\right\}, \quad \forall \alpha \in A, \forall \psi \in C^2(\overline{\Omega}),$$

where $D\psi$ denotes the gradient and $D^2\psi$ the Hessian matrix of ψ. The coefficients b, σ, f and c are defined on $\overline{\Omega} \times A$ and take their values in $\mathbb{R}^d$, $\mathbb{R}^d \otimes \mathbb{R}^p$ (with $p \geq 1$), $\mathbb{R}$ and $]0, +\infty[$ respectively. We also denote by $\sigma^*(x,\alpha)$ the transposed matrix of $\sigma(x,\alpha)$. Finally, the boundary condition φ is a real-valued function defined on $\partial\Omega$.

We provide an extension of the "strong comparison result" (i.e. a comparison type result for semicontinuous viscosity solutions [2]), proven in Barles and Rouy [5] theorem 2.1 p. 2001 (see also Barles and Burdeau [4]) to the case of domains with a non-smooth boundary, more precisely to the case of domains with irregularities pointing outwards the domain (the typical example being an intersection of regular open sets). This kind of result is a key argument to establish that the value function of a stochastic exit time control problem is continuous and that it is the unique viscosity solution of the associated Bellman-Dirichlet problem. It is also used to prove the convergence of approximation schemes (cf. Barles, Souganidis [6]). The case when the boundary $\partial\Omega$ is regular has been deeply studied, for different kinds of boundary conditions (Dirichlet [4], [5], Neumann [2], state constraints [9]).

An interesting application of this extension is a strong comparison result for parabolic problems, on cylindrical domains (such as $\Omega =]0,T[\times Q$, which is not smooth considered as a subset of $\mathbb{R}^d$). Usually, such a system is proved to have a unique solution in a given class of continuous functions (see for example [3]), our result allows to conclude to uniqueness and continuity of the solution in a class of discontinous functions.

Remark 1. Since we shall always use the notion of viscosity solutions here, we will drop the term "viscosity" in the whole sequel and simply refer to subsolutions, supersolutions and solutions. We do not recall the definition of these objects which can be found in [4] definition 1.1 p. 136 for instance.

2. The strong comparison result

We make the following assumptions :

(H1) Ω is an open bounded subset of $\mathbb{R}^d$ and A is a compact separable metric space. The functions σ, b, c and f are continuous on $\overline{\Omega} \times A$. For any $\alpha \in A$, $\sigma(.,\alpha)$ and $b(.,\alpha)$ are Lipschitz continuous functions on $\overline{\Omega}$, moreover

$$\sup_{\alpha \in A} ||\phi(.,\alpha)||_{C^{0,1}(\overline{\Omega})} < \infty,$$

for $\phi = \sigma_{i,j}, b_i$ $(1 \leq i \leq d, 1 \leq j \leq p)$.

(H2) $c > 0$ on $\overline{\Omega} \times A$.

(H3) $\varphi \in C(\partial\Omega)$.

(H4) For any $x \in \partial\Omega$, the set

$$Z(x) = \left\{\zeta \in C^2(\mathbb{R}^d) \mid \zeta(x) = 0,\ \zeta > 0 \text{ in } \Omega \text{ and } D\zeta(x) \neq 0\right\}$$

[2] See Crandall, Ishii, Lions [8] and Barles [1] for a presentation of the notion of viscosity solution.

is nonempty.

Note that **(H4)** holds under the exterior ball condition :

$$\forall x \in \partial\Omega,\ \exists y \in I\!R^d \setminus \{0\} \text{ such that } \overline{B(x+y,|y|)} \cap \overline{\Omega} = \{x\},$$

where $B(x,r)$ denotes the ball with center x and radius r.

Let us note d the distance to the boundary $\partial\Omega$, i.e. $d(x) = \inf_{y\in\partial\Omega} |x-y|$, for all $x \in \overline{\Omega}$. Following the notations of [5], we now introduce :

$$\Gamma_{in} = \left\{ x \in \partial\Omega \;\middle|\; \begin{array}{c} d \text{ is } C^2 \text{ in a neighborhood of } x \text{ and} \\ \forall \alpha \in A, \sigma^*(x,\alpha)Dd(x) = 0 \text{ and } L^\alpha d(x) \geq 0 \end{array} \right\},$$

$$\text{and } \Gamma_{out} = \left\{ x \in \partial\Omega \;\middle|\; \begin{array}{c} \exists \zeta \in Z(x) \text{ such that} \\ \forall \alpha \in A, \sigma^*(x,\alpha)D\zeta(x) \neq 0 \text{ or } L^\alpha \zeta(x) < 0 \end{array} \right\}.$$

(H5) Γ_{in} is an open subset of $\partial\Omega$ and $\Gamma_{in} \cup \Gamma_{out} = \partial\Omega$.

Γ_{in} is defined as in in [5], it is a subset of the smooth part of $\partial\Omega$. The definition of Γ_{out} is slightly more general and applies to non-smooth boundaries. This allows us to weaken the assumptions of theorem 2.1 p. 2001 in [5], Γ_{in} and Γ_{out} may now be connected.

Remark 2. **(H5)** implies that Γ_{out} is a closed subset of $\partial\Omega$.

Remark 3. If $\partial\Omega$ is $W^{3,\infty}$, as in [5], then Γ_{in} is a closed subset of the boundary by its very definition. So, in this case, **(H5)** implies that Γ_{in} (and thus Γ_{out}) is a union of connected components of $\partial\Omega$. This means that, in our case, $\overline{\Gamma_{in}} \cap \overline{\Gamma_{out}}$ is a subset of the non-smooth part of $\partial\Omega$.

Theorem 2.1 (Strong Comparison Result). *Assume that* **(H1)-(H5)** *hold. If u (resp. v) is a subsolution (resp. supersolution) of (1), then*

$$u \leq v \text{ in } \Omega.$$

Remark 4. **(H5)** may be weakened. Indeed, it can be replaced by

(H5') Γ_{in} is an open subset of $\partial\Omega$ and $\Gamma = \partial\Omega \setminus (\Gamma_{in} \cup \Gamma_{out})$ is an open subset of $\partial\Omega$ satisfying **(H6)** p. 2000 in [5].

Example 1 (Intersection of regular open sets). Assume that the bounded open set Ω can be written as the intersection of two open sets Ω_1 and Ω_2 such that the distance d_i to the boundary $\partial\Omega_i$ is a C^2 function in a neighborhood of this boundary, for $i = 1$ and 2. If we define the sets

$$\Gamma^i_{out} = \{x \in \partial\Omega_i \mid \forall \alpha \in A, \sigma^*(x,\alpha)Dd_i(x) \neq 0 \text{ or } L^\alpha d_i(x) < 0\} \text{ for } i = 1 \text{ and } 2,$$

we have

$$x \in \left(\Gamma^1_{out} \cup \Gamma^2_{out}\right) \cap \partial\Omega \Longrightarrow x \in \Gamma_{out}.$$

Indeed, if $x \in \partial\Omega_i$ for some i then d_i obviously belongs to $Z(x)$. In particular, if $x \in \partial\Omega_1 \cap \partial\Omega_2$ (the non-smooth part of $\partial\Omega$), then d_1 and d_2 belong to $Z(x)$.

Note also that this theorem applies to the case of non-countable intersections, like a cone in $I\!R^3$.

Proof of theorem 2.1. The proof is similar to theorem 2.1 p. 2001 in [5]. Roughly speaking, it consists in considering a maximum point x_0 of $u - v$ on $\overline{\Omega}$ and obtain a contradiction by assuming $u(x_0) - v(x_0) > 0$. Under the assumptions of the theorem, this maximum point is either in the open set $\Omega \cup \Gamma_{in}$ (and then the contradiction is given by proposition 4.2 p. 2006 in [5]) or in Γ_{out}. In the second case, we conclude using the following result :

Proposition 2.1. *Assume that* **(H1)-(H5)** *hold. If u (resp. v) is a subsolution (resp. supersolution) of (1), then*

$$u \le \varphi \le v \text{ on } \Gamma_{out},$$

i.e. the Dirichlet boundary condition holds in the classical sense on Γ_{out}.

Proof of proposition 2.1. The proof is similar to the regular case (proposition 4.1 p.2006 in [5], see also proposition 1.1 p.140 in [4]), the only idea is to replace systematically the distance d to the boundary by a function $\zeta \in Z(x_0)$, for each $x_0 \in \Gamma_{out}$.

3. Application to parabolic problems

Let $T > 0$ and let Q be an open bounded subset of $I\!R^n$ (with $n \in I\!N$). We now consider the following parabolic equation with an initial condition for $t = 0$ and a transversal boundary condition for $t \in]0, T[$:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t, y) + \sup\limits_{\alpha \in A} \{-\Lambda^\alpha u(t, y) + c(t, y, \alpha)u(t, y) - f(t, y, \alpha)\} = 0 & \text{in }]0, T[\times Q, \\ u(0, y) = \varphi(0, y) & \text{on } \overline{Q} \\ u(t, y) = \varphi(t, y) & \text{on }]0, T[\times \partial Q. \end{cases} \tag{2}$$

For each $t_0 \in [0, T]$, $\Lambda^\alpha(t_0, .)$ is defined by

$$\Lambda^\alpha \psi(t_0, y) = B(t_0, y, \alpha).D\psi(y) + \frac{1}{2}\text{trace}\left\{S(t_0, y, \alpha)S^*(t_0, y, \alpha)D^2\psi(y)\right\}, \quad \forall \alpha \in A, \forall \psi \in C^2\left(\overline{Q}\right)).$$

Note that this equation may be written in the form (1), by setting

$$d = n + 1,\ \Omega =]0, T[\times Q,\ x = (t, y),\ b = \begin{pmatrix} -1 \\ B \end{pmatrix} \text{ and } \sigma = \begin{pmatrix} 0 \ldots 0 \\ S \end{pmatrix},$$

and considering that φ is defined on $\partial\Omega$ (i.e. even on the terminal boundary $\{T\} \times Q$).

Corollary 3.1. (Strong Comparison Result for parabolic problems). *Assume that* **(H1)-(H3)** *hold. Let δ be the distance to the boundary ∂Q, assume δ is C^2 in a neighborhood of this boundary in Q and*

$$\forall t \in]0, T],\ \forall y \in \partial Q,\ \forall \alpha \in A,\ S^*(t, y, \alpha)D\delta(y) \ne 0 \text{ or } \Lambda^\alpha\delta(y) < 0. \tag{3}$$

If u (resp. v) is a subsolution (resp. supersolution) of (2), then

$$u \le v \text{ in }]0, T[\times Q.$$

Remark 5. By classical arguments, **(H2)** may be replaced in this corollary by

(H2') $c > \lambda$ on $\overline{\Omega} \times A$, for some $\lambda \in I\!R$.

Proof of corollary 3.1. **(H4)** is straightforward, since Ω may be written as the intersection of the smooth domains $]0, T[\times I\!R^n$ and $I\!R \times Q$, so we just have to check **(H5)**.

Let us note d_0 the distance to the closed initial boundary $\left(\{0\} \times \overline{Q}\right)$, we have $d_0(t,y) = t$ and therefore

$$L^\alpha d_0(t,y) = -\frac{\partial d_0(t,y)}{\partial t} + \Lambda^\alpha d_0(t,y) = -1 \text{ on } \left(\{0\} \times \overline{Q}\right), \text{ for any } \alpha \in A.$$

Using example 1, this implies $\left(\{0\} \times \overline{Q}\right) \subset \Gamma_{out}$. For the transversal boundary, using again the notations of section 2 (see also example 1), assumption (3) reads $(]0,T] \times \partial Q) \subset \Gamma_{out}$. In a neighborhood of the open terminal boundary $(\{T\} \times Q)$, the distance d to $\partial\Omega$ satisfies $d(t,y) = T - t$ (it is thus C^2), so we have

$$S^*(t,y,\alpha)Dd(t,y) = 0 \text{ and } -\frac{\partial d(t,y)}{\partial t} + \Lambda^\alpha d(t,y) = 1 > 0 \text{ on } (\{T\} \times Q), \text{ for any } \alpha \in A,$$

i.e. $(\{T\} \times Q) \subset \Gamma_{in}$.

It is now clear that **(H5)** holds, since $\Gamma_{in} = (\{T\} \times Q)$ is an open subset of $\partial\Omega$ and $\Gamma_{in} \cup \Gamma_{out} = \left(\{0\} \times \overline{Q}\right) \cup (]0,T] \times \partial Q) \cup (\{T\} \times Q) = \partial\Omega$.

Remark 6. This result may be extended to more general domains. Assume that Ω is the intersection

$$\Omega = (]0,T[\times \mathbb{R}^n) \cap \left(\bigcap_{i=1}^{q} \Omega_i\right),$$

where for all $i \in \{1,...,q\}$, Ω_i is an open subset of $\mathbb{R}^{n+1}$, such that the distance d_i to the boundary $\partial\Omega_i$ is C^2 in a neighborhood of this boundary in Ω_i, and $\frac{\partial d_i}{\partial t} \neq 0$ on $\partial\Omega_i$ for $t \in [0,T]$. Corollary 3.1 still holds with assumption (3) replaced by : for all $(t,y) \in \partial\Omega$, for some $i \in \{1,...,q\}$ such that $(t,y) \in \partial\Omega_i$,

$$\forall \alpha \in A,\ S^*(t,y,\alpha)Dd_i(t,y) \neq 0 \text{ or } -\frac{\partial d_i(t,y)}{\partial t} + \Lambda^\alpha d_i(t,y) < 0. \tag{4}$$

References

[1] G. Barles, *Solutions de Viscosité des Equations de Hamilton-Jacobi.* Mathématiques et Applications 17, Springer Verlag (1994). 2

[2] G. Barles, *Nonlinear Neumann Boundary Conditions for Quasilinear Elliptic Equations and Applications.* J. Differential Equations, Vol. 154, No. 1 (1999), pages 191-224. 2

[3] G. Barles, R. Buckdahn and E. Pardoux, *Backward Stochastic Differential Equations and Integral-Partial Differential Equations.* Stochastics and Stochastics Reports, Vol. 60 (1997), pages 57-83. 2

[4] G. Barles and J. Burdeau, *The Dirichlet Problem for Semilinear Second-Order Degenerate Elliptic Equations and Applications to Stochastic Exit Time Control Problems.* Communications in Partial Differential Equations, Vol. 20, No. 1&2 (1995), pages 129-178. 2, 2, 2, 4

[5] G. Barles and E. Rouy, *A Strong Comparison Result for the Bellman Equation Arising in Stochastic Exit Time Control Problems and its Applications.* Communications in Partial Differential Equations, Vol. 23, No. 11&12 (1998), pages 1945-2033. 2, 2, 3, 3, 3, 3, 3, 3, 4, 4

[6] G. Barles and P. E. Souganidis, *Convergence of Approximation Schemes for Fully Nonlinear Second Order Equations.* Asymptotic Analysis, Vol. 4, Elsevier Science Publishers B.V., North-Holland (1991), pages 273-283. 2

[7] S. Chaumont, *Gestion optimale de bilan de compagnie d'assurance.* PhD Thesis, Université Nancy I (2002).

[8] M.G. Crandall and H. Ishii and P.-L. Lions, *User's guide to Viscosity Solutions of 2nd Order PDE.* Bulletin of the American Mathematical Society, Vol. 27, No. 1 (1992), pages 1-67. 2

[9] M. A. Katsoulakis, *Viscosity Solutions of Second Order Fully Nonlinear Elliptic Equations with State Constraints.* Indiana University Mathematics Journal, Vol. 43, No. 2 (1994), pages 493-517. 2

[10] N. V. Krylov, *Controlled Diffusion Processes.* Springer Verlag, Berlin(1980). 1

[11] P.-L. Lions, *Optimal Control of diffusion processes and HJB equations part II : Viscosity Solutions and Uniqueness.* Communications in Partial Differential Equations, Vol. 8 (1983), pages 1229-1276. 1